Using the Schwarzchild Metric to Describe an Object Orbiting Black Hole

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1 Theory

When investigating and describing the movement of an object in the space around a black hole, it has become evident that we cannot treat gravity as a force exactly but as a warping of the space and time around it. In lecture, we described the manifold (the set of points attached to the space), the coordinates (to label the points) and the metric (the way to measure distance). It turns out that General Relativity, which is a set of rules about how space acts with respect to the coordinates and invariant quantities that the metric constructs, can be solved with a number of metrics. I will be focusing on the Schwarzchild Metric:

$$ds^2 = -(1 - \frac{r_s}{r})c^2 dt^2 + \frac{dr^2}{(1 - \frac{r_s}{r})} + r^2(d\theta^2 + sin^2(\theta)d\phi^2) \quad (1)$$

The differential components each represent something different: ds represents the 'spacetime distance', which can be used to find proper length; dt represents time, dr represents radial motion, and $d\phi$ and $d\phi$ represent the motion in the theta and phi directions, working in spherical coordinates. In this project, I will work entirely in the plane where $\theta = \pi/2$, so the $d\theta$ term goes to zero.

I also needed the equations for constants of motion, which describe how quantities like linear momentum, angular momentum, and energy are related to the differential quantities over the differential proper time. The equations of motion I used were:

$$\frac{E}{m} = (1 - \frac{r_s}{r})c^2 \frac{dt}{d\tau} \quad (2)$$

$$L = mr^2 \frac{d\phi}{d\tau} \quad (3)$$

τ represents the proper time, or 'stopwatch time' of the moving object, just like ds represents the proper length that an object travels. ds and $d\tau$ are related like $d\tau^2 = -\frac{ds^2}{c^2}$.

2 Process and Equations

1. I began by using the structure of Homework 3 problem 4b that asked us to find the circular orbits by setting the derivative of the effective potential with respect to r equal to zero, where:

$$\frac{dV_{eff}}{dr} = -\frac{L^2}{mr^3} + \frac{GMm}{r^2} + \frac{3L^2GM}{r^4mc^2} \tag{4}$$

   Setting this equal to zero, I got an equation for my angular momentum at each of those orbitals:

$$L = rc\sqrt{\frac{GMm}{rc^2 - 3GM}} \tag{5}$$

2. After this, I found through trial and error that I would need an equation representing the differential equation of L with respect to r.

$$\frac{dL}{dr} = c\sqrt{\frac{GMm}{rc^2 - 3GM}}(1 - \frac{rc^2}{rc^2 - 3GM}) \tag{6}$$

3. Now I needed to ask what the energy of my object at each of these orbitals would be. In order to do this, I took the Schwarzchild metric described in Equation 1, and set my dr=0 because I was finding the energy at each orbital, hence why I felt I could make this assumption. By dividing through the metric by $d\tau^2$ and substituting in constants of motion, I got an equation for energy in terms of angular momentum and radius.

$$E = c\sqrt{m^2c^2 - \frac{m^2c^2r_s}{r} + \frac{L^2}{r^2} - \frac{L^2r_s}{r^3}} \tag{7}$$

4. In the same vein as for the angular momentum, I found the equation for $\frac{dE}{dr}$.

$$\frac{dE}{dr} = \frac{c}{2r^{5/2}} \frac{m^2c^2r^2r_s - 2L^2r + 3L^2r_s + \frac{dL}{dr}2Lr^2 - \frac{dL}{dr}2Lrr_s}{\sqrt{m^2c^2r^3 - m^2c^2r_sr^2 + L^2r - L^2r_s}} \tag{8}$$

5. Using the equation given to us in Homework 4 Problem 2, we know that the change of radius with respect to time is $\frac{-\frac{dE}{dr}}{L_GW}$, where we have found the value for $\frac{dE}{dr}$ and the formula for $L_{GW} = -\frac{32}{5}\frac{G^4}{c^5}\frac{(M+m)M^2m^2}{r^5}$, which represents the luminosity emanated by gravitational waves.

6. We also have the constants of motion equations that give us a formula for $\frac{d\phi}{d\tau}$, which we can then turn into an equation I can plug into my computer to solve numerically. Because my $\frac{dr}{dt}$ equation was with respect to t, I wanted to get the $\frac{d\phi}{d\tau}$ into a differential with respect to t, but when I found that differential I couldn't plug it into my computer numerically

because I didn't have a way to evaluate at a different set r at each time step, so instead I put $\frac{d\phi}{d\tau}$ in terms of r, which looks (horrendous) something like this:

$$(\frac{L_{GW}}{\frac{dE}{dr}})(\frac{c^2 m(1-\frac{r_s}{r})}{E})(\frac{c}{mr}\sqrt{\frac{GMm}{rc^2-3GM}}) \tag{9}$$

7. Using `solve_ivp()`, I plugged the differential equations into my computer to solve numerically. As sanity checks, I needed to plot the radii and then the radii vs angle before I tried the animation, which gave me a good result.

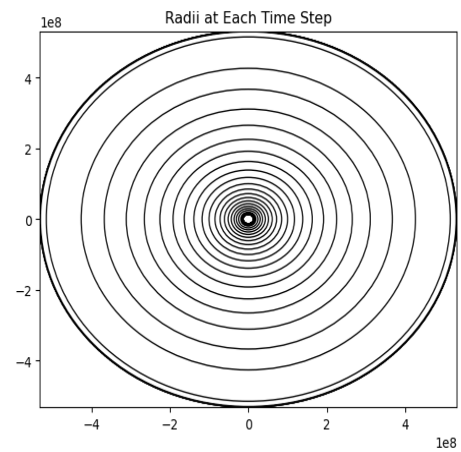

Figure 1: Radii at Each Time Step

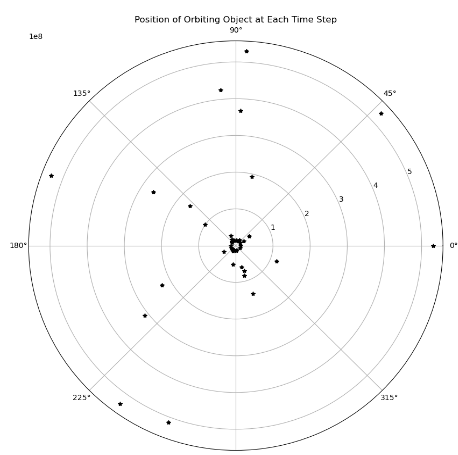

Figure 2: Positions at Each Time Step

8. The only thing left to do was to animate, and play around with different sizes of time steps! I chose to use Cartesian coordinates because I was going to try playing with simulating 2 bodies of comparable mass, but this was out of scope.

Also included in my submission is my python files and the animation I created as the final deliverable for this project.

3 Future Directions and Reflection

In the future, I thought I would like to try working on animating a binary black hole merger, but having investigated it I found that doing this is one of the great unsolved mysteries for GR, and I was pretty sure I wouldn't be solving that for my Astronomy 161 final project. That being said, I learned a lot about the assumptions that I have to make and equations I have to use in order to get operable differential equations that can describe this behavior. One strange thing about my animation is that the radius seems to asymptote at about $6.62r_s$, which is not a number I recognized in the slightest. In future, I would like to investigate the source of this, as it may have been an issue with an assumption I made, or even just a silly math error.